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DETECTION OF FLUCTUATING SONAR TARGETS

By
R. J. Urick
G. C. Gaunaurd

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13 ABSTRACT

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WHITE OAK, SILVER SPRING, Md.

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Detection of Flucturating Sonar Targets

This report concerns the effect of fluctuations on the detectability of active and passive sonar targets. It will be of interest to those concerned with sonar detection in the real-world environment.

The work was done under a project entitled "Underwater Sound Fluctuation", SHIP-16604/SF 52-552-008.

ROBERT WILLIAMSON II
Captain, USN
Commander

Z. I. SLAWSKY
By direction

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INTRODUCTION

1. In sonar prediction, it is often required to estimate the probability that a given target can be detected at a given range. To do this, the standard procedure is to find the output signal-to-noise-ratio by adding up the various pertinent sonar parameters, and then to apply one of a set of theoretical curves to obtain the probability of detection P_D for an acceptable false alarm rate P_{FA}. These curves are the well-known receiver operating characteristic curves, or ROC curves, that are valid for a steady signal in a Gaussian noise background. With this approach, a curve like the solid curve of Fig. 1A is obtained if P_D is plotted against range. P_D is near unity at short ranges and falls rapidly with range to near zero at long ranges.

2. However, when real field data are examined, it becomes clear that the actual P_D falls off more slowly with range, like the dashed curve of Fig. 1A. The two curves cross in the vicinity of $P_D = 0.5$ if the mean values of the parameters have been chosen correctly. P_D is lower than it ought to be at short ranges and is higher at long ranges. An example of a real P_D vs. range curve as obtained in a fleet exercise is shown in Fig. 1B. With field data such an effect may be caused in part by system factors or field procedures. Examples are loss of target at short ranges due to vertical directionality of the receiving array or to overloading by an excessive signal. Another contributing factor is the manner in which the field data is reduced to obtain P_D. Yet in many exercises it is clear that the gradual variation of P_D with range is real, and that the ROC-curves from which it was determined are incorrect when used in the real world. It is suggested here that this effect is due to fluctuations in the signal and in the mean value of the noise background.

3. The existence of such fluctuations is well-known to anyone with experience in underwater sound. Indeed, an outstanding feature of the subject is that nothing is ever constant for any length of time. Transmitted signals, and to a lesser extent the background in which they appear, have non-stationary statistics, with a broad spectrum of temporal fluctuations ranging from the very slow to the very fast.

4. It is easy to see why fluctuations can cause P_D to fall off more slowly with range than it would in their absence. At long range, detection occurs when the signal is strong, while at short ranges, the target is lost, and detection fails to occur, when the signal is weak. Since P_D cannot be greater than unity nor less than zero, the effect is to increase P_D at long ranges and to decrease it at short ranges.

5. The aim of this report is to examine the effect of fluctuations on the ROC-curves and to present some samples of new ROC-curves that include fluctuations and thereby become more applicable to the real ocean. The new ROC curves give the single-look detection probability and false alarm probability for a given average signal-to-noise ratio when Gaussian signal fluctuations of a given magnitude are present.

THE PRESENT MODEL

6. The subject of detection of fluctuating targets has received much attention in the radar literature, notably by Marcum (1) (2) and Swerling (3) (4). Various probability density functions were considered, and ROC curves were derived for pulse-to-pulse and scan-to-scan fluctuations of radar echoes and noise using processing systems having small bandwidth-time products. However, in sonar it is both customary and efficient from a detection standpoint to use processors having large BT products (compared to unity) for which, by the Central Limit Theorem, the output fluctuations must have a Gaussian amplitude distribution, regardless of the distribution of signal and noise at the processor input. Accordingly, in our model the output fluctuations will be assumed to be Gaussian. For fluctuations fast compared to the signal observation time, Gaussian distributions will be adopted for both signal and noise. For fluctuations slow compared to the signal observation time, the mean amplitude of the signal during one observation time or "look" will be taken to vary in a Gaussian manner about an ensemble mean. For both slow and fast fluctuations, the ROC curves will be found similar in shape to ordinary ROC curves, but with a slope that will depend upon the standard deviation of the fluctuations.

DETECTION WITH AND WITHOUT FLUCTUATIONS

7. Detection theory involves the setting of a threshold at the output of the processing-display system. A crossing of this threshold indicates a detection. This crossing may be produced by the signal, in which case the detection is valid and a true detection occurs. Alternatively, a crossing may be produced by noise alone and may occur in the absence of a signal, in which case a false-alarm occurs. The probability of these occurrences depends upon the amplitude distribution of noise and signal-plus-noise at the receiver output. The area under these distribution or probability density curves, to the right of the selectable threshold, is equal to P_{FA} and P_D, respectively, when the total area under the curves is normalized to unity.

8. In sonar, signal fluctuations are caused by variations in the ocean medium as well as, in echo ranging, by ping-to-ping changes of aspect of the target. In radar, only the latter are important. These signal fluctuations have a wide spectral range from the very fast to the very slow, from the nearly instantaneous changes caused by interferences between propagation paths to the long-term changes brought about by seasonal variability of the oceans.

9. However, while we will allow signal fluctuations of these kinds to occur, we will at the same time assume that the mean noise background remains constant. This is equivalent to saying that any changes in the background - such as would be caused by a changing ship speed or sea state - will be assumed to be compensated for by an adjustment in threshold setting. These adjustments can be made manually by an alert observer, or automatically through suitable adaptive techniques.

10. Figure 2 illustrates the effect of fluctuations on the distribution curves of noise N and signal-plus-noise S+N. The case of no fluctuations is shown in Fig. 2A. After detection, the signal remains flat-topped, and all the fluctuations in S+N are due to the noise. The distribution, or probability density, curves of N and S+N are Gaussian with the same standard deviation ($\sigma_{S+N} = \sigma_N$), but with the curve for S+N shifted to the right by an amount proportional to the amplitude of the signal.

1. The corresponding curves when fluctuations are present are shown in Fig. 2B and Fig. 2C. Fig. 2B shows the case of fast fluctuations, that is, fluctuations more rapid than the signal duration, such that fluctuations exist in both the signal and noise background within the observation or "look" time of the signal. Now the amplitude distribution curve of S+N is shifted to the right of that for N alone and, in addition, has a larger standard deviation ($\sigma_{S+N} > \sigma_N$). On the other hand, with slow fluctuations relative to the signal duration shown in Fig. 2C, the signal may be regarded as being constant during any one look. The mean values of the distribution curves of S+N will be taken to vary in a Gaussian way from look-to-look about some ensemble average. Two examples are shown in Fig. 2C. A weak signal yields a distribution curve lying to the left of the ensemble average $(S+N)_e$, with $\sigma_{S+N} = \sigma_N$; a strong signal has a distribution lying to the right of $(\overline{S+N})_e$, again with $\sigma_{S+N} = \sigma_N$. The fluctuation is described by a certain standard deviation σ_M of the mean look values about the ensemble mean $(\overline{S+N})_e$.

12. In Appendix A it is shown that the ROC curves with either fast or slow fluctuations present are given in parametric form by the expressions

$$P_D = 1/2 \text{ erfc} \left[\frac{1}{\sqrt{2k}} \; \frac{T}{\sigma_N} - d^{1/2} \right]$$

$$P_{FA} = 1/2 \text{ erfc} \left[\frac{T}{\sqrt{2}\sigma_N} \right]$$

where erfc X is the complementary error function defined by

$$\text{erfc } X = 1-\text{erf } X; \ \text{erf } X = \frac{2}{\sqrt{\pi}} \int_0^X e^{-\tau^2} d\tau.$$

T is a parameter representing a threshold setting, σ_N is the standard deviation of the noise; $d^{1/2} = \frac{\overline{(S+N)}_e}{\sigma_N}$, and k is a fluctuation index defined by

$$k = \frac{\sigma_{S+N}}{\sigma_N} = \sqrt{1+\sigma_S^2/\sigma_N^2}$$

where σ_S is the standard deviation of the signal. For normal ROC's for no fluctuations, k = 1. The error function erf x has been extensively tabulated (5) (6).

13. We conclude from the above expressions and the properties of the complementary error function that

(1) the ROC's with Gaussian fluctuations are straight lines in double probability coordinates, as are normal ROC's;

(2) all the ROC's pass through the point $P_D = 1/2$, independent of the fluctuations index k;

(3) when fluctuations are present (k>1) the ROC's have a smaller slope than normal ROC's to a degree depending on the fluctuation index k.

EXAMPLES OF FLUCTUATION-MODIFIED ROC CURVES

14. Figure 3 shows ROC curves for three values of d with fluctuations absent (k=1) and with fluctuations present (k=2). It will be observed that at a constant P_{FA} the effect of fluctuations is to increase P_D with P_D < 50% and to decrease P_D when P_D > 50%. Thus, fluctuations improve detection at low signal-to-noise ratios and degrade it at high signal-to-noise ratios. Fluctuation-modified ROC's are also given in Fig. 4 for four values of k for the single value d=16. When the magnitude of the fluctuation is extreme (k=10), the detection probability remains near 50% for all values of d and for all threshold settings; the effect of varying the threshold setting is to change P_{FA} without appreciably affecting P_D. The larger value of k, the larger the increase in d that is required to produce a given increase of P_D at constant P_{FA}.

15. Another way of plotting ROC curves is shown in Fig. 5 where P_{FA} is held constant at 10^{-4} and the abscissa is $d^{1/2}$ and 10 log $d^{1/2}$. With this kind of plot the effect of fluctuations in improving P_D for

weak signals and in degrading it for strong signals becomes readily apparent.

OUTPUT VS INPUT FLUCTUATIONS

16. The fluctuation index k refers to the fluctuations occurring at the output of the processor at the place where thresholding is done. Its magnitude depends upon both the fluctuations of signal and noise occurring in the ocean (i.e. at the input of the processing system) as well as upon the characteristics of sensor and processing employed. For example, for Gaussian noise fed into a square-law detector followed by an integrator it can be shown (7) that for BT>>1,

$$(\sigma_N)_o = (\sigma_N)_{i/4BT}$$

where $(\sigma_N)_i$ is the standard deviation of the noise at the input of the detector and $(\sigma_N)_o$ is that at the output; B is the input bandwidth and T is the integration time of the integrator. While the integrator reduces the fluctuations of the noise, it will not affect signal fluctuations slow compared to the integration time. In any case, as mentioned above, the output fluctuations will be Gaussian, regardless of the distribution at the input, as long as many samples of signal and noise are summed up by the integrator (i.e. BT>>1).

OBSERVED FLUCTUATION OF UNDERWATER SOUND SIGNALS

17. Present knowledge of the fluctuations of sound in the sea, and the value of k to use in realistic circumstances is extremely small. What has received most attention theoretically and experimentally have been the pulse-to-pulse fluctuations of short high frequency pulses traveling over relatively short distances via either the direct path from source to receiver, or via the surface reflection. Such studies are of primary interest to torpedo-homing. Here the fluctuation is due to the thermal micro-structure of the medium, to the roughness of the sea surface, or to both together. In the literature it is customary to measure the fluctuation in terms of the coefficient of variation of amplitude V, equal to the standard deviation of a series of pressure amplitudes divided by the mean amplitude. Values of V have been found (8) (9) (10) to range from 0.05 to 0.25 at 1000 yds., with an increase as the square-root of the range; the maximum possible value of V for a Rayleigh distribution theoretically amounts to 0.52. Fluctuations of this kind, due to inhomogenities in the medium and on its boundaries, would be considered long for the short pulselengths of active homing torpedoes, but would be considered short for most other applications.*

*The coefficient of variation V is related to the fluctuation index k by $k^2 = 1+d^2V^2$, where d is the ROC-curve parameter.

13. Another kind of fluctuation is due to multi-path interference. The simplest example is interference between the direct path and the surface reflection. More important are interferences between the multipaths occurring in long range duct propagation, such as those in the mixed layer duct, the SOFAR channel, and in shallow water. Two examples of this sort of fluctuation that have appeared in the literature are tidally-produced changes of 10-20 db with a 12-hour period in shallow water transmission (11) and fluctuations with periods of the order of 1 minutes over transmission paths from Eleuthera to Bermuda (12).

14. For other cases of sonar interest, little or no knowledge exists. An important subject for research investigation concerns the amplitude and time scale of the fluctuations of signals received in the sea under the conditions of sonar interest.

SUMMARY

15. In the presence of fluctuations, the standard ROC curves do not give valid estimates of the detection probability as a function of signal-to-noise ratio. When Gaussian signal fluctuations occur, the ROC curves (that had been straight lines on double probability paper in absence of fluctuations) become straight lines of smaller slope, so as to indicate that a larger change in signal-to-noise ratio is needed to produce a given change in detection probability. The effect of fluctuations is to increase the detection probability at low S/N ratios and to decrease it at high S/N ratios. Unfortunately, the magnitude and time-scale of signal and noise fluctuations existing in the real ocean are all but unknown. Obtaining quantitative data on the magnitude of signal and noise fluctuations under realistic sonar circumstances is a prime research problem for the future.

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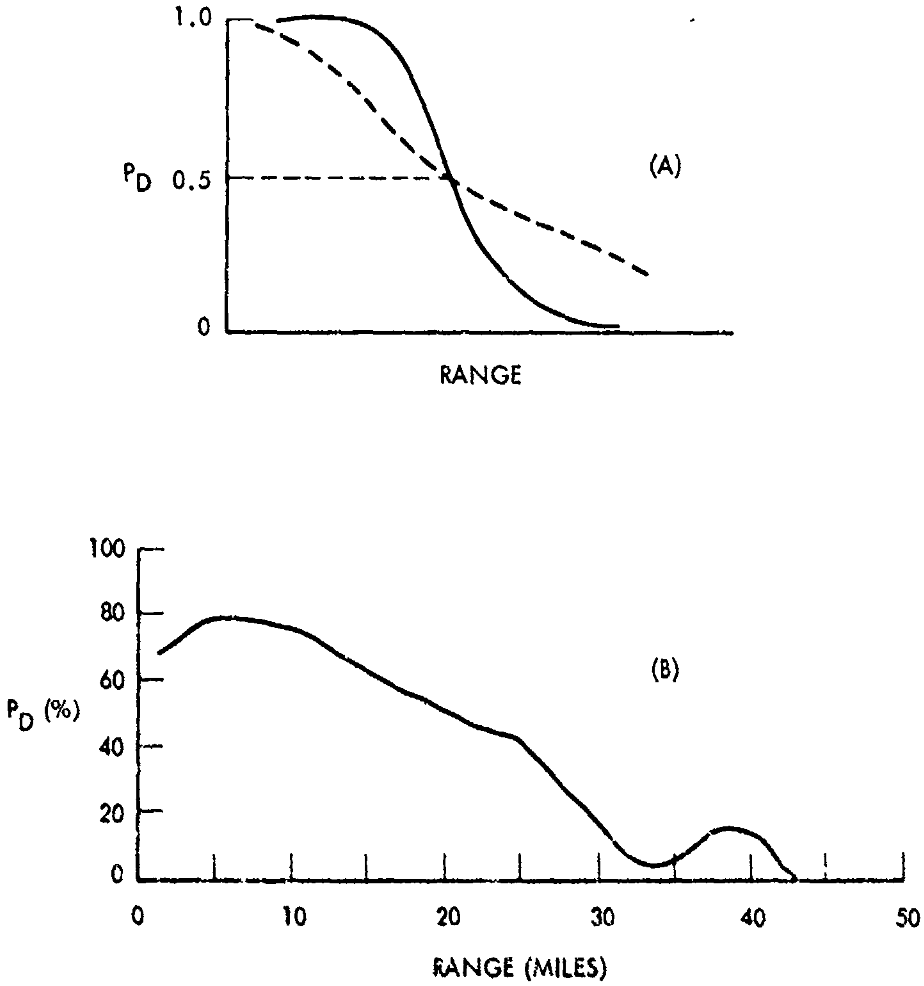

FIG. 1 (A) P(D) AS A FUNCTION OF RANGE AS CALCULATED (SOLID CURVE) AND AS OBSERVED (DASHED CURVE), (B) EXAMPLE FROM A FLEET EXERCISE.

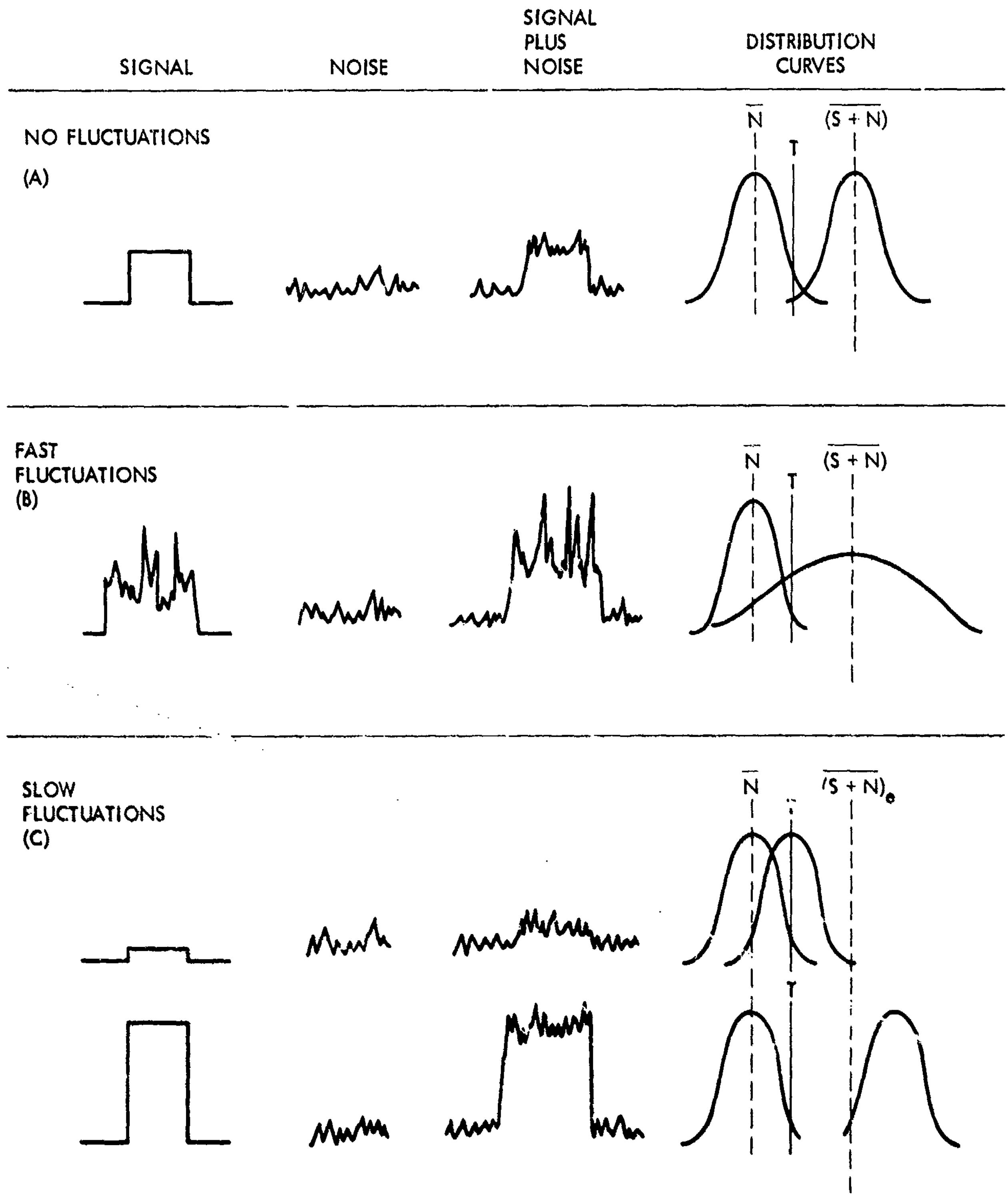

FIG. 2 WAVEFORM AND AMPLITUDE DISTRIBUTION OF FLUCTUATING SIGNALS AND NOISE.

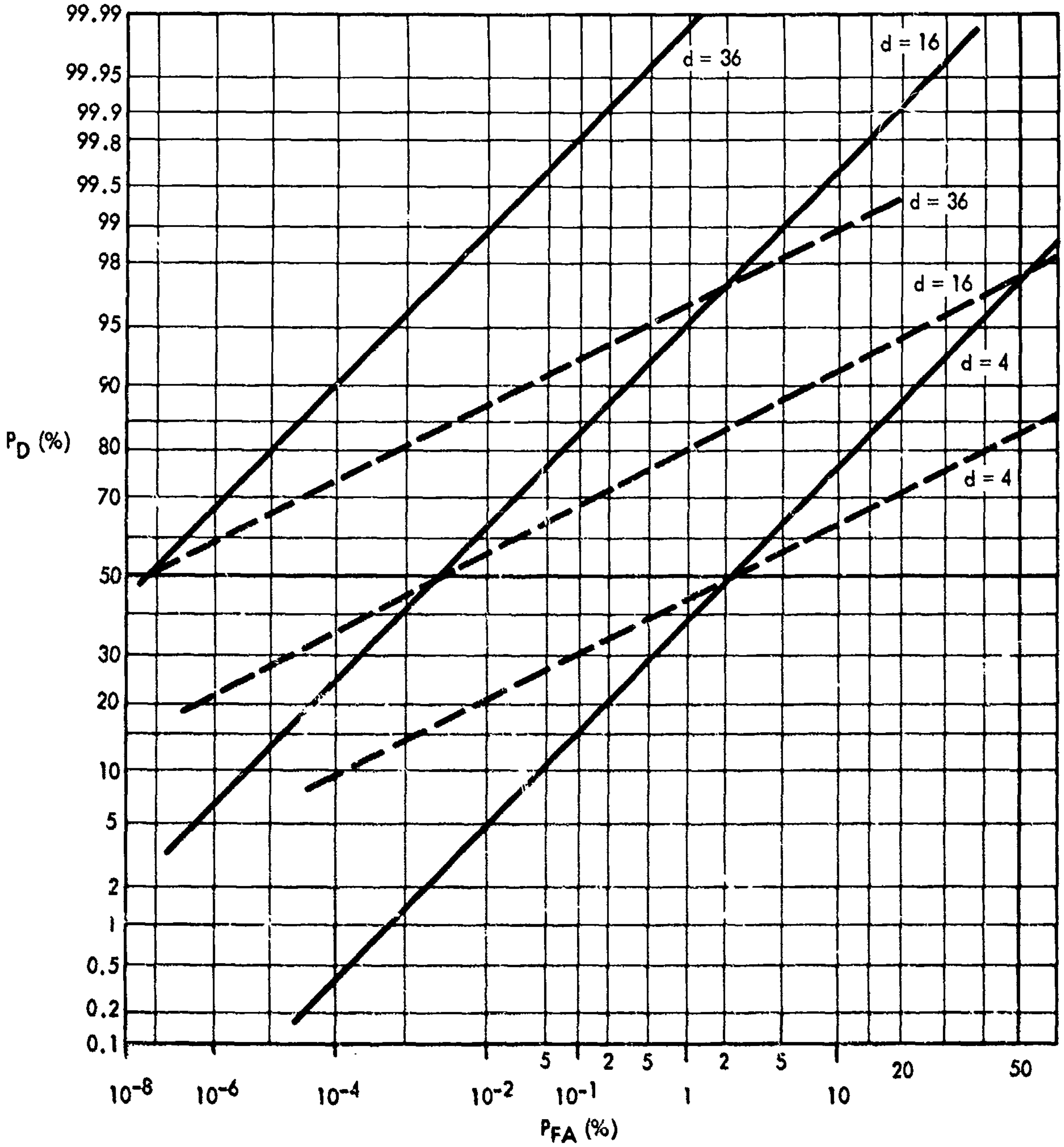

FIG. 3 ROC CURVES FOR VARIOUS VALUES OF d WITH AND WITHOUT FLUCTUATIONS PRESENT. SOLID LINES, NO FLUCTUATIONS, k = 1. DASHED LINES, WITH FLUCTUATIONS, k = 2.

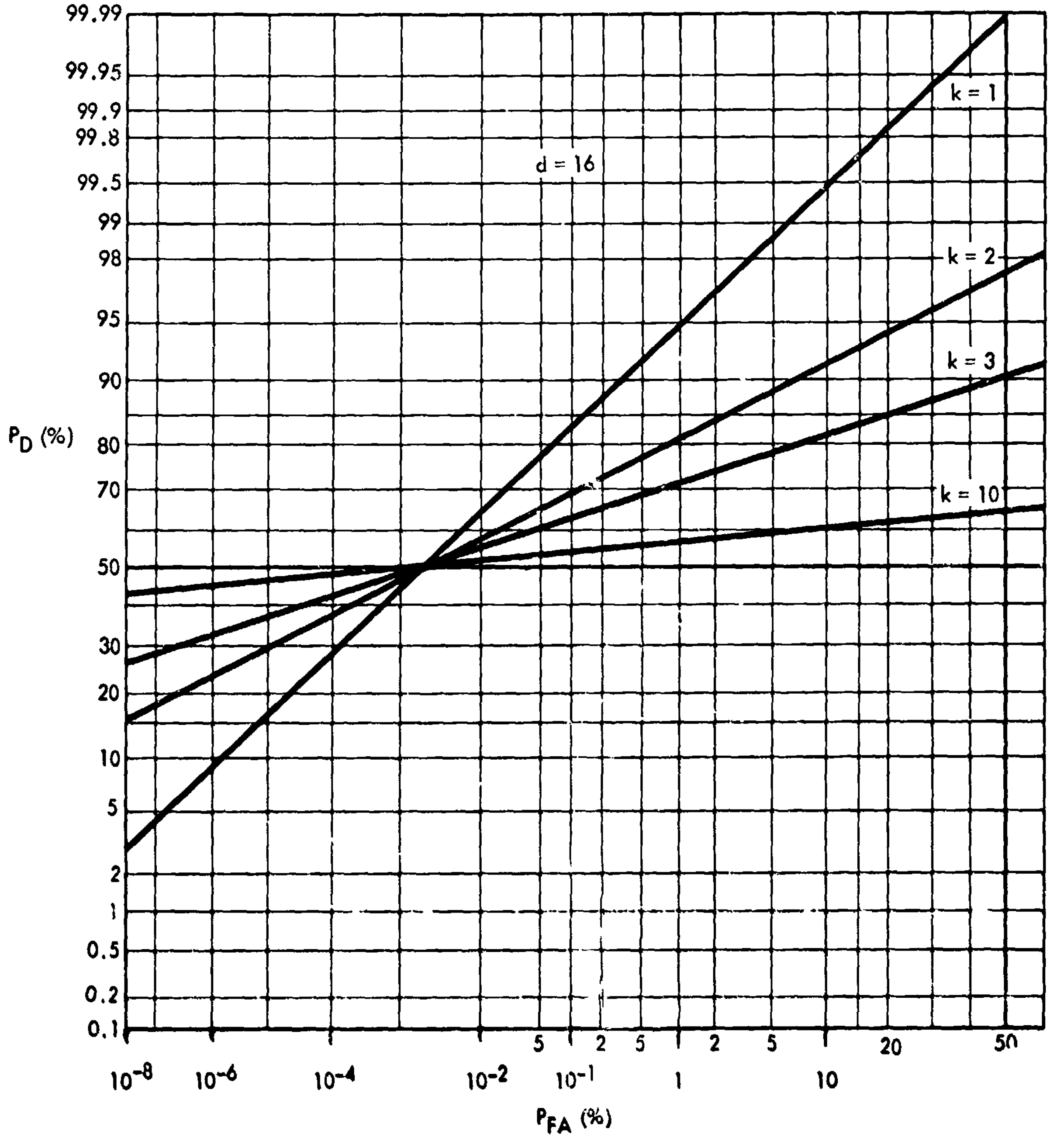

FIG. 4 ROC CURVES FOR VARIOUS VALUES OF THE FLUCTUATION PARAMETER k

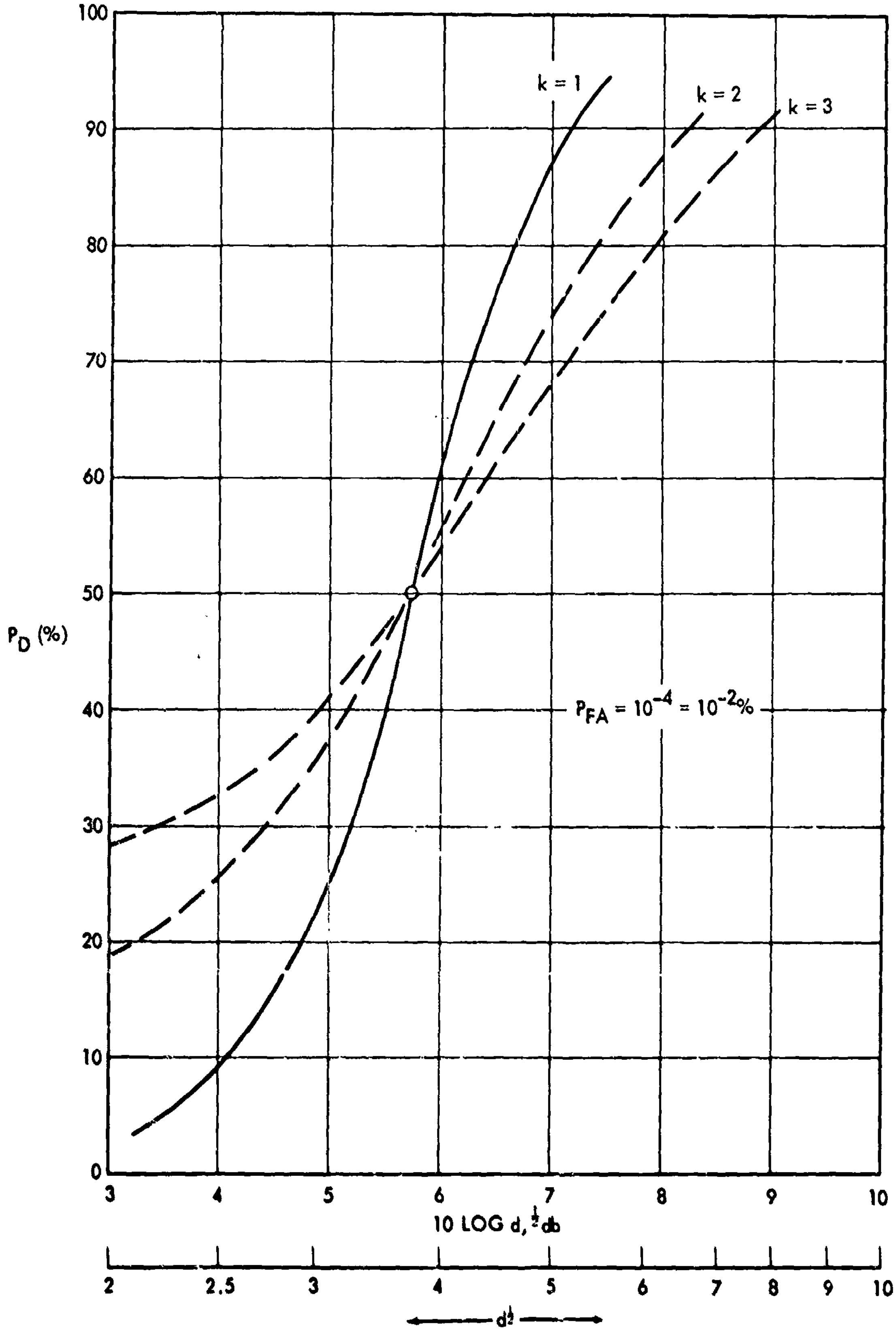

FIG. 5 ROC CURVES FOR A CONSTANT FALSE ALARM RATE.

APPENDIX A

BASIC THEORY WITH GAUSSIAN STATISTICS

Let the probability density of the signal be $f_N(x)$ and that for signal plus noise, $f_{S+N}(x)$. Both density functions are assumed to be Gaussian, with means $\overline{N}$ and $\overline{S+N}$ and standard deviations σ_N and σ_{S+N}. That is, we assumed

$$f_N(x) = \frac{1}{\sqrt{2}\pi\sigma_N} e^{-1/2\left(\frac{x-\overline{N}}{\sigma_N}\right)^2}$$

$$f_{S+N}(x) = \frac{1}{\sqrt{2}\pi\sigma_{S+N}} e^{-1/2\left(\frac{x-(\overline{S+N})}{\sigma_{S+N}}\right)^2} \tag{1}$$

This assumption of Gaussian statistics is a realistic one. For, no matter what the statistics of signal and noise may be at the input to the processor, the output will be, by the Central Limit Theorem, Gaussian as long as the system bandwidth B and the integration time T are large enough for many data samples to be integrated, that is for BT>>1.

The ROC curves are defined by

$$P_D = \int_{x=T}^{\infty} f_{S+N}(x)\, dx$$

$$P_{FA} = \int_{x=T}^{\infty} f_N(x)\, dx \tag{2}$$

where P_D and P_{FA} are the probabilities of detection and a false alarm, and T is the output amplitude threshold setting. With the Gaussian functions for f_N and f_{S+N}, these integrals can be evaluated by means of the complementary error function

$$\text{erfc } x = 1 - \text{erf } x$$

$$\text{erf } T = \frac{2}{\sqrt{\pi}} \int_0^T e^{-x^2}\, dx \tag{3}$$

for which tables are readily available. Substituting (1) and (2) and using (3) we find

$$P_D = 1/2 \text{ erfc}\left[\frac{T - \overline{(S+N)}}{\sqrt{2}\,\sigma_{S+N}}\right]$$

$$P_{FA} = 1/2 \text{ erfc}\left[\frac{T - \overline{N}}{\sqrt{2}\,\sigma_N}\right] \tag{4}$$

NO FLUCTUATIONS

For the case of a non-fluctuating signal, we must have

$$\sigma_{S+N} = \sigma_N$$

so that from (4)

$$P_D = 1/2 \text{ erfc}\left[\frac{T - \overline{(S+N)}}{\sqrt{2}\,\sigma_N}\right]$$

$$P_{FA} = 1/2 \text{ erfc}\left[\frac{T - \overline{N}}{\sqrt{2}\,\sigma_N}\right]$$

It is convenient to define a parameter called the <u>detection index</u>, d, such that

$$d^{1/2} = \frac{\overline{S+N} - \overline{N}}{\sigma_N}$$

the ratio of the mean of the signal at the output to the standard deviation of the noise. Without loss of generality, and for simplicity's sake, we can take $\overline{N} = 0$, and so obtain

$$P_D = 1/2 \text{ erfc}\left[\frac{1}{\sqrt{2}} \left(\frac{T}{\sigma_N} - d^{1/2}\right)\right]$$

$$P_{FA} = 1/2 \text{ erfc}\left[\frac{1}{\sqrt{2}} \frac{T}{\sigma_N}\right] \tag{5}$$

These are the equations of the ordinary ROC curves for the case of a non-fluctuating signal. For each selected value of $d^{1/2}$, a ROC curve can be generated by varying the normalized threshold ratio T/σ_N.

FAST FLUCTUATIONS

When fast signal fluctuations, relative to the time constant RC of the integrator, are present in addition to the fluctuations of the noise, $\sigma_{S+N} \neq \sigma_N$. The extent of the fluctuations can be specified by a parameter k defined by

$$k = \frac{\sigma_{S+N}}{\sigma_N}$$

Equation (4) now becomes

$$P_D = 1/2 \text{ erfc}\left[\frac{T - (\overline{S+N})}{\sqrt{2}\, k\sigma_N}\right]$$

$$P_{FA} = 1/2 \text{ erfc}\left[\frac{T - \overline{N}}{\sqrt{2}\,\sigma_N}\right]$$

On substituting $d^{1/2} = \dfrac{\overline{S+N} - \overline{N}}{\sigma_N}$ and taking $\overline{N} = 0$ as before, we obtain

$$P_D = 1/2 \text{ erfc}\left\{\frac{1}{\sqrt{2}\, k}\left[\frac{T}{\sigma_N} - d^{1/2}\right]\right\}$$

$$P_{FA} = 1/2 \text{ erfc}\left[\frac{1}{\sqrt{2}}\frac{T}{\sigma_N}\right] \tag{6}$$

These are the parametric equations of the ROC curves in terms of the two parameters $d^{1/2}$ and k. Equations (6) reduce to (5) when k = 1.

SLOW FLUCTUATIONS

The case of slow fluctuations leads to an identical result when the parameter k is suitably defined. In this case the signal can be taken to be essentially constant during any one "look" observation time, so that $\sigma_{S+N} = \sigma_N$ as in the case of no fluctuations. However, the mean amplitude $\overline{S+N}$ during one look will be taken to be distributed in a Gaussian manner, with standard deviation σ_M, about some emsemble mean amplitude $(\overline{S+N})_e$. The effective single look probability of detection in the presence of slow fluctuations may be regarded as being that for a single look of mean $\overline{S+N}$, multiplied by the probability of

occurrence of $\overline{S+N}$ about the ensemble mean $(\overline{S+N})_e$, and integrated over all values of $\overline{S+N}$. That is to say, writing $\overline{S+N} = M$ and $(\overline{S+N})_e = M_e$ for brevity, the present model yields

$$(P_D)_e = \int_{-\infty}^{+\infty} P(M,M_e) \cdot P_D(T,M)\, dM$$

where $(P_D)_e$ is the effective detection probability when fluctuations are present, $P(M,M_e)$ is the probability density of M about M_e, and $P_D(T,M)$ is the detection probability at threshold setting T when the single-look mean amplitude is M. If the probability density of M is Gaussian, we may write

$$P(M,M_e) = \frac{1}{\sqrt{2\pi}\ \sigma_M} \cdot \exp\left(-1/2(M - M_e)^2/\sigma_M{}^2\right)$$

where σ_M is the standard deviation of the slow fluctuations. Combining with the appropriate expression for detection probability

$$P_D(T,M) = \int_T^{\infty} \frac{1}{\sqrt{2\pi}\ \sigma_N} \exp\left[-1/2\left(\frac{x-M}{\sigma_N}\right)^2\right] dx$$

we have,

$$P_D(M,T) = \int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi}\ \sigma_M} \exp(-1/2(M-M_e)^2/\sigma_M{}^2) \cdot \int_T^{\infty} \frac{1}{\sqrt{2\pi}\ \sigma_N} \exp\left(-1/2\ (x-M)^2/\sigma_N{}^2\right)dx\right] dM \tag{7}$$

It is shown in Appendix B that this latter expression, when $\bar{N} = 0$ as before, becomes

$$(P_D)_e = 1/2 \text{ erfc}\left[\frac{1}{\sqrt{2}\ k_1}\left(\frac{T}{\sigma_N} - d_1{}^{1/2}\right)\right]$$

where

$$d_1{}^{1/2} = \frac{M_e}{\sigma_N} = \frac{(\overline{S+N})_e}{\sigma_N}$$

and

$$k = \frac{\sqrt{\sigma_M{}^2 + \sigma_N{}^2}}{\sigma_N}$$

The result is the same as for the fast fluctuations except that $d^{1/2}$ is defined in terms of the ensemble mean amplitude M_e and the fluctuation factor k is defined in terms of the standard deviation σ_M of the slow fluctuations about the ensemble mean M_e. For both fast and slow fluctuations of the signal, P_{FA} is the same as for no

fluctuations, inasmuch as the statistics of the noise background are assumed to be constant.

APPENDIX B

EVALUATION OF AN INTEGRAL

Equation (7) of Appendix A can be written as

$$(p_D)_{eff} = \int_{x=T}^{\infty} dx \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_N\sigma_M} \exp\left[-\frac{1}{2}\left\{\left(\frac{x-M}{\sigma_N}\right)^2 + \left(\frac{M-M_e}{\sigma_M}\right)^2\right\}\right] dM$$

$$= \int_{x=T}^{\infty} dx\, f_{\xi_1}(x)$$

where

$$f_{\xi_1}(x) = K \int_{-\infty}^{+\infty} e^{-Q}\, dM$$

with $K = \frac{1}{2\pi\sigma_N\sigma_M}$, $Q = \frac{(x-M)^2}{2\sigma_N^2} + \frac{(M-M_e)^2}{2\sigma_M^2}$

It is clear that $K e^{-Q}$ is the joint density of x & M and its integral over M, viz., $f_{\xi_1}(x)$, is the unconditional probability of x. This integral can be performed by writing Q in a more convenient way as

$$Q = \frac{(\sigma_N^2 + \sigma_M^2)}{2\sigma_N^2\sigma_M^2}\left[M - \frac{x\sigma_M^2 + M_e\sigma_N^2}{\sigma_N^2 + \sigma_M^2}\right]^2 + \frac{(x - M_e)^2}{2(\sigma_N^2 + \sigma_M^2)}$$

The change of variables

$$v = M - \frac{x\sigma_M^{\ 2} + M_e\sigma_N^{\ 2}}{\sigma_N^{\ 2} + \sigma_M^{\ 2}} \quad , \qquad dv = dM$$

yields

$$f_{\xi_1}(x) = \frac{1}{2\pi\sigma_N\sigma_M} \, e^{-\frac{1}{2}\left[\frac{x - M_e}{\sqrt{\sigma_N^{\ 2} + \sigma_M^{\ 2}}}\right]^2} \int_{-\infty}^{+\infty} e^{-a^2v^2} \, dv$$

where $$a = \frac{\sqrt{\sigma_N^{\ 2} + \sigma_M^{\ 2}}}{\sqrt{2\pi}\ \sigma_N\sigma_M}$$

and since the integral above can be handled by the gamma function i.e.,

$$\int_{-\infty}^{+\infty} e^{-a^2v^2} \, dv = \frac{\sqrt{\pi}}{a}$$

we find,

$$f_{\xi_1}(x) = \frac{1}{\sqrt{2\pi}\ \sqrt{\sigma_N^{\ 2} + \sigma_M^{\ 2}}} \, e^{-\frac{1}{2}\left\{\frac{x - M_e}{\sqrt{\sigma_N^{\ 2} + \sigma_M^{\ 2}}}\right\}^2}$$

which is another Gaussian density centered about the ensemble mean with variance equal to the sum of the variances.

The effective detection probability with these fluctuations taken into account is thus,

$$(P_D)_{eff} = \int_{x=T}^{\infty} f_{z_1}(x)dx = \int_{x=T}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_N^2 + \sigma_M^2}} e^{-\frac{1}{2}\left[\frac{x-M_e}{\sqrt{\sigma_N^2+\sigma_M^2}}\right]^2} dx$$

$$= \frac{1}{2} \operatorname{erfc}\left[\frac{T - M_e}{\sqrt{2}\sqrt{\sigma_N^2 + \sigma_M^2}}\right]$$

Introducing the definitions,

$$k_1 = \frac{\sqrt{\sigma_N^2 + \sigma_M^2}}{\sigma_N} \qquad d'^{1/2} = \frac{M_e - \bar{N}}{\sigma_N}$$

The above reduces to

$$(P_D)_{eff} = \frac{1}{2} \operatorname{erfc}\left[\frac{T - (\bar{N} + \sigma_N\ d'^{1/2})}{\sqrt{2}\ k_1\ \sigma_N}\right]$$

which in the case of noise of zero mean (i.e., $\bar{N} = 0$) reduces to the result

$$(P_D)_{eff} = \frac{1}{2} \operatorname{erfc}\left[\frac{T - \sigma_N\ d'^{1/2}}{\sqrt{2}\ k_1\ \sigma_N}\right]$$